Taylor Series

1. Def:
   - Assume that f is a function of class C^∞ that can be represented by a power series near some point $x=a$.

$$f(x) = f(a) + C_1(x-a) + C_2(x-a)^2 + \dots + C_n(x-a)^n + \dots$$
$$|x-a|<R$$

If we differentiate $f(x)$ continuously, we get

1. $f'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \dots$
   Note: $f'(a) = C_1 = 1! \cdot C_1$

2. $f''(x) = 2C_2 + (2\cdot3)C_3(x-a) + (3\cdot4)C_4(x-a)^2 + \dots$
   Note: $f''(a) = 2C_2 = 2! \cdot C_2$

3. $f'''(x) = (2\cdot3)C_3 + (2\cdot3\cdot4)C_4(x-a) + \dots$
   Note: $f'''(a) = (2\cdot3)C_3 = 3! \cdot C_3$

The pattern is:

$$f^{(n)}(a) = n! \cdot C_n \longrightarrow C_n = \frac{f^{(n)}(a)}{n!}$$

- Thm:
  If $f \in C^\infty$ has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n \text{ for } |x-a|<R,$$

then its coefficients are given by the formula

$$C_n = \frac{f^{(n)}(a)}{n!}$$

I.e. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$= f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$

This is called the Taylor Series of the function f at a.

- Maclaurin Series:
  If $a=0$, then the function is called the Maclaurin Series.

  I.e. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

  $= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$

2. Partial Sum of Taylor Series:
   - Denoted by $T_n(x)$.

   - $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$

     $= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$

   - $T_n(x)$ is a polynominal of degree n called the n^{th}-degree Taylor Polynomial of f at a, or the n^{th}-order Taylor Polynomial of f at a.

— Note:

1. $T_n(x)$ is a polynomial in powers of $x-a$ rather than in powers of x.

2. $T_1(x) = f(a) + \frac{f'(a)}{1!}(x-a)$ is the line tangent to $y=f(x)$ at the point $(a, f(a))$. $y=T_1(x)$ and $y=f(x)$ have the same slope $f'(a)$ at this point.

3. The 2nd derivative measures the way the curve $y=f(x)$ is bending as it passes through $(a, f(a))$. That is, $f''(a)$ is the concavity of $y=f(x)$ at $(a, f(a))$. $y=T_2(x)$ and $y=f(x)$ have the same concavity, $f''(a)$, at this point.

4. $y=T_3(x)$ and $y=f(x)$ will have the same rate of change of concavity at $(a, f(a))$.

5. The larger n is, the more closely the n^{th} degree Taylor Polynomial will approx $f(x)$ for x near a.

T_1 is linear approx.
T_2 is quad approx.

- If $f(x)$ is the sum of the Taylor Series, then $f(x) = \lim_{n \to \infty} T_n(x)$.

- $R_n(x) = f(x) - T_n(x)$. It is the error made if $f(x)$ is replaced with the approximation $T_n(x)$. $R_n(x)$ is called the n^{th} degree remainder for $f(x)$ at $x=a$.

- Thm:
  If $f(x) = T_n(x) + R_n(x)$, where T_n is the n^{th}-degree Taylor Polynomial of f at a and $\lim_{n \to \infty} R_n(x) = 0$ for

  $|x-a| < R$, then f is equal to the sum of its Taylor Series on the interval $|x-a| < R$.

3. Examples:

   1. Directly compute the second degree Taylor Polynomial about the point $(\pi, 4)$ for $f(x,y) = \tan\left(\frac{x}{y}\right)$.

      Soln:

      $$f(\pi,4) = \tan\left(\frac{\pi}{4}\right)$$
      $$= 1$$

      $$f_x(\pi,4) = \left.\frac{\sec^2\left(\frac{x}{y}\right)}{y}\right|_{(\pi,4)} = \frac{1}{2}$$

      $$f_y(\pi,4) = -\left.\frac{x\sec^2\left(\frac{x}{y}\right)}{y^2}\right|_{(\pi,4)} = -\frac{\pi}{8}$$

$$f_{xx}(\pi,4) = \frac{2\sec^2\left(\frac{x}{y}\right)\tan\left(\frac{x}{y}\right)}{y^2}\Bigg|_{(\pi,4)}$$

$$= \frac{1}{4}$$

$$f_{xy}(\pi,4) = \frac{-2x\sec^2\left(\frac{x}{y}\right)\tan\left(\frac{x}{y}\right) - y\sec^2\left(\frac{x}{y}\right)}{y^3}\Bigg|_{(\pi,4)}$$

$$= -\left(\frac{\pi+2}{16}\right)$$

$$f_{yy}(\pi,4) = \frac{x\left(2x\sec^2\left(\frac{x}{y}\right)\tan\left(\frac{x}{y}\right) + 2y\sec^2\left(\frac{x}{y}\right)\right)}{y^4}\Bigg|_{(\pi,4)}$$

$$= \frac{\pi^2+4\pi}{64}$$

$$T_2 = f(\pi,4) + f_x(\pi,4)(x-\pi) + f_y(\pi,4)(y-4) + \left(\frac{1}{2}\right)\left(f_{xx}(\pi,4)\right)(x-\pi)^2 + \left(f_{xy}(\pi,4)\right)(x-\pi)(y-4) + \left(\frac{1}{2}\right)\left(f_{yy}(\pi,4)\right)(y-4)^2$$

$$= 1 + \left(\frac{1}{2}\right)(x-\pi) + \left(-\frac{\pi}{8}\right)(y-4) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)(x-\pi)^2 + \left(\frac{-\pi-2}{16}\right)(x-\pi)(y-4) + \left(\frac{1}{2}\right)\left(\frac{\pi^2+4\pi}{64}\right)(y-4)^2$$

2. Write the first four terms of the Taylor Series for $f(x,y) = \ln(1+x+y)$.

Soln:

We know that $\ln(1+s) = s - \frac{s^2}{2} + \frac{s^3}{3} - \frac{s^4}{4} \ldots$

Subbing $(x+y)$ for s, we get

$$(x+y) - \frac{(x+y)^2}{2} + \frac{(x+y)^3}{3} - \frac{(x+y)^4}{4}$$ as

the first four terms.

3. Give the fourth-order Taylor polynomial for the function $f(x,y) = \cos(xy)$ around the point $(x,y) = (0,0)$.

Soln:

We know that $\cos(r) = 1 - \frac{r^2}{2} + \frac{r^4}{4} - \ldots$.

Subbing in xy for r, we get

$$\cos(xy) = 1 - \frac{(xy)^2}{2} + \ldots$$

Note: Since the question asked for the 4^{th}-order, we go up until we list all the terms of degree 4 or less. Here, 1 has a degree of 0 and x^2y^2 has a degree of 4. The next term has a degree >4, so we don't include it.

The first 3 series converge for all $x \in \mathbb{R}$.

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

2. $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x^{2n+1}) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

3. $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x^{2n}) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

These 3 series converge if $|x| < 1$.

4. $(1 + x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x + \ldots$

5. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$

6. $\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

This is the Geometric Series.

7. $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots$

   This converges to $\frac{a}{1-r}$ if $|r| < 1$.